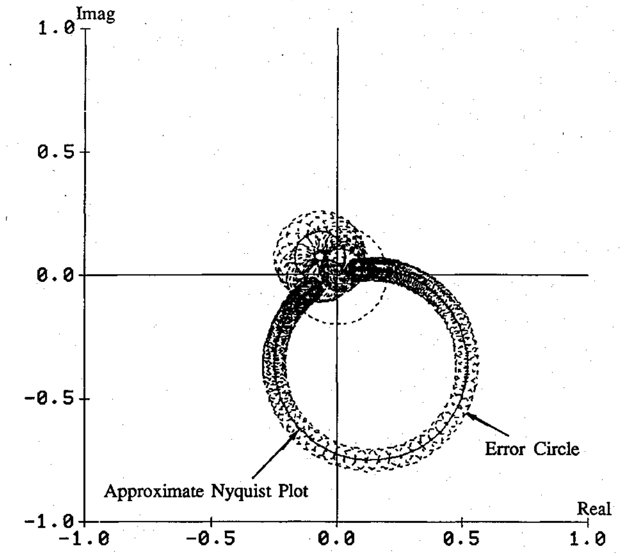

Fig. 3 Approximate Nyquist plot with error circles.[9]

Nyquist plots of the approximate system and the error bounds. For several illustrative examples see Chait et al.[2] and Chait.[9]

Numerical Example

Consider a transfer function of the form of Eq. (1) where $\omega_k = (k\pi)^2$, $\delta = 2$, $\epsilon_1 = 0.005$, and $\epsilon_2 = 0.5$. This transfer function corresponds to the ratio between a position point sensor to a point actuator for the Bernoulli-Euler beam with unity parameters.[2,9]

The uniform bound R_1 and the frequency dependent bound $R_2(\omega)$ can be calculated using Eqs. (4) and (5) for $\sigma_0 = 0$. For this system, we have $\rho = 2$, $\delta = 2$, and $\Gamma_1 = \Gamma_2 = 0.01$. For a truncated series that consists of the first term alone ($n = 1$):

$$R_1 = 200\sum_{k=2}^{\infty}\frac{1}{(k\pi)^4} \approx 0.1692$$

$$R_2(\omega) = (200/\omega)\sum_{k=2}^{\infty}\frac{1}{(k\pi)^2} \approx 13.16/\omega$$

for $n = 2$, $R_1 = 0.041$ and $R_2(\omega) = 8.106/\omega$; for $n = 4$, $R_1 = 0.0075$ and $R_2(\omega) = 4.59/\omega$; and for $n = 10$, $R_1 = 0.00000373$ and $R_2(\omega) = 2.02/\omega$.

For the transfer function considered in this example with shifted imaginary axis by $\sigma_0 = 0.1$, similar bounds can be computed using Eqs. (4) and (5). For $n = 1$, since $\zeta_2\omega_2 > 0.1$, the range of $\tilde{\zeta}_k$ is found to be $\tilde{\epsilon}_1 = 0.00246$ and $\tilde{\epsilon}_2 = 0.497$. Thus, the bounds for $n = 1$ are $R_1 = 0.344$ and $R_2(\omega) = 26.79/\omega$. For $n = 10$, since $\zeta_{11}\omega_{11} > 0.1$, we compute $\tilde{\epsilon}_1 = 0.0049$ and $\tilde{\epsilon}_2 = 0.497$, and the modified bounds are $R_1 = 0.00000381$ and $R_2(\omega) = 2.06/\omega$. Note that the bounds become larger as the shifted axis is closer to the $(n+1)$th root. Consequently, the closer the desired closed-loop decay rate is to the decay rate of the open-loop modes, the more costly is the control solution.

Conclusions

Several useful formulas were developed for computing approximation error bounds for certain models of large space structures. These are the uniform and frequency dependent bounds about the imaginary axis and about a shifted imaginary axis. It was shown that these bounds can capture arbitrary variations of the damping factor within a specified range and allow uncertainties in the location of the actuators and sensors. In addition, these bounds can be computed along any vertical axis in the open left half-plane, which is necessary for control synthesis for a specified decay rate. They can be employed, in robust control techniques, to obtain safe estimates of closed-loop frequency responses corresponding to systems that have either infinite degrees of freedom or a high-order model. Numerical computation aspects and graphical interpretation are discussed.

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Orbital Motion Under Continuous Radial Thrust

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Introduction

In general, the problem of orbital motion of spacecraft under the disturbing influence of continuous thrust produced by rocket propulsion requires numerical methods for solution. However, for a vehicle initially in a nearly circular orbit, when the thrust acceleration is constant in magnitude and directed either radially or tangentially, some analytical results of practical interest may be obtained.[1-4] A comprehensive analysis of orbital motion under continuous low thrust is presented by Battin[1,2] from numerical solutions and from analytic solutions originally given by Tsien,[3] for the case of constant radial acceleration, and by Benney,[4] for the case of constant tangential acceleration.

The analytic solution obtained for the problem of constant radial acceleration being applied to a vehicle in circular orbit reveals an interesting result. It is found that there is a critical value of constant radial acceleration above which escape speed will eventually be attained and below which the vehicle will simply spiral out to a higher altitude and then return, with continuation of thrust, to the initial altitude. This analytic

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solution for constant radial acceleration is similar to, but slightly different from, the one presented herein for the case of constant specific thrust acceleration (i.e., constant ratio of radial acceleration to the varying acceleration of gravity). The two solutions provide essentially the same results for small thrust levels and time intervals such that the spiral flight path does not extend far beyond the initial orbit. However, when this does happen and the acceleration of gravity decreases appreciably, the two solutions depart significantly. Since the ratio of thrust to weight (based on the vehicle mass and the ambient acceleration of gravity) remains fixed in the present case and increases continually in the other, there is a difference in the unwinding spiral paths to reach either maximum altitude or escape speed. Moreover, the critical ratio of constant radial acceleration to the acceleration of gravity is different from the critical value of thrust-to-weight ratio found in the present case. Furthermore, a critical radial distance (for escape speed to be attained in) of five times that of the initial orbit was found in the case of constant radial acceleration but not in the present case.

The purpose of this Note is to show how a solution may readily be obtained for the effects of continuous radial thrust when the thrust is a fixed fraction of the varying vehicle weight in orbit. The solution is valid for any fixed value of thrust-to-weight ratio and indicates that escape speed is eventually attained when this ratio is greater than one to two. In addition to a description of the motion, the solution provides time-of-flight information and approximate changes in vehicle mass due to expenditure of rocket propellant. To document the solution, tabulated results are presented for several values of thrust-to-weight ratio where escape speed is reached and where it is not. As a practical application of the results, the problem of interplanetary transfer from the orbit of Earth to that of Mars, using solar radiation pressure as a means of propulsion,[5] is a case in point.

Equations of Motion

When thrust of magnitude T is directed at angle τ to the flight path, the equations expressing the balance of forces in and normal to the direction of motion, respectively, are

$$\frac{dV}{dt} = \left(\frac{T}{m}\right) \cos \tau - g \sin \gamma \tag{1}$$

$$\frac{V d\gamma}{dt} = \left(\frac{T}{m}\right) \sin \tau + \left(\frac{V^2}{r} - g\right) \cos \gamma \tag{2}$$

where V is the vehicle velocity, m its mass, t is time, γ the flight-path angle, r is radial distance from the planet's center, and g the acceleration of gravity. These equations may be expressed more conveniently in dimensionless form as

$$\sqrt{r/g}\,\frac{d\bar{V}}{dt} = \left(\frac{T}{mg}\right) \cos \tau - \left(1 - \frac{\bar{V}^2}{2}\right) \sin \gamma \tag{3}$$

$$\sqrt{r/g}\,\frac{\bar{V} d\gamma}{dt} = \left(\frac{T}{mg}\right) \sin \tau - (1 - \bar{V}^2) \cos \gamma \tag{4}$$

where $\bar{V}$ is the vehicle velocity divided by the circular orbital velocity u_c, as given by

$$\bar{V} \equiv V/u_c = V/\sqrt{gr} = V/\sqrt{\mu/r} \tag{5}$$

with μ being the gravitational parameter.

In the case of thrust in the radial direction, $\tau = 90\text{ deg} - \gamma$ so that $\cos\tau = \sin\gamma$ and $\sin\tau = \cos\gamma$. Thus, Eqs. (3) and (4) become

$$\sqrt{r/g}\,\frac{d\bar{V}}{dt} = \left[\frac{T}{mg} - 1 + \bar{V}^2/2\right] \sin \gamma \tag{6}$$

$$\sqrt{r/g}\,\frac{\bar{V} d\gamma}{dt} = \left[\frac{T}{mg} - 1 + \bar{V}^2\right] \cos \gamma \tag{7}$$

From Eq. (6), using $dr/dt = \bar{V}\sqrt{gr} \sin \gamma$, it is found that

$$\frac{dr}{r} = 2\bar{V}\, d\bar{V} \Big/ \left[2\left(\frac{T}{mg} - 1\right) + \bar{V}^2\right] \tag{8}$$

which integrates into

$$\frac{r_2}{r_1} = \left[2\left(\frac{T}{mg} - 1\right) + \bar{V}_2^2\right] \Big/ \left[2\left(\frac{T}{mg} - 1\right) + \bar{V}_1^2\right] \tag{9}$$

and so, for $\bar{V}_1 = 1$ and $\bar{V}_2 = \sqrt{2}$,

$$\frac{r_2}{r_1} = \left(\frac{T}{mg}\right) \Big/ \left(\frac{T}{mg} - \frac{1}{2}\right) \tag{10}$$

Thus, in order to attain escape speed ($\bar{V} = \sqrt{2}$) by thrusting radially from a circular orbit with T/mg fixed, T/mg must be greater than ½.

Dividing Eq. (7) by Eq. (6) gives

$$\tan\gamma\, d\gamma = 2\left[\left(\frac{T}{mg} - 1\right) \Big/ \bar{V} + \bar{V}\right] d\bar{V} \Big/ \left[2\left(\frac{T}{mg} - 1\right) + \bar{V}^2\right] \tag{11}$$

which integrates into

$$\ln\left(\frac{\cos\gamma_1}{\cos\gamma_2}\right) = \frac{1}{2} \ln \left\{\bar{V}^2\left[2\left(\frac{T}{mg} - 1\right) + \bar{V}^2\right]\right\}\Bigg|_{\bar{V}_1}^{\bar{V}_2} + \ln\left[2\left(\frac{T}{mg} - 1\right) + \bar{V}^2\right]\Bigg|_{\bar{V}_1}^{\bar{V}_2} \tag{12}$$

so that finally

$$\cos\gamma_2/\cos\gamma_1 = \frac{\bar{V}_1}{\bar{V}_2}\left[\frac{2(T/mg - 1) + \bar{V}_1^2}{2(T/mg - 1) + \bar{V}_2^2}\right]^{1/2} \tag{13}$$

or, using Eq. (9),

$$\cos\gamma_2/\cos\gamma_1 = (\bar{V}_1/\bar{V}_2)(r_1/r_2)^{1/2} \tag{14}$$

From Eq. (13), with $\bar{V}_1 = 1$ and $\gamma_1 = \gamma_2 = 0$, it is found that

$$\bar{V}_2^2 = (1 - T/mg)\{1 - [1 - (1 - 2T/mg)/(1 - T/mg)^2]^{1/2}\} \tag{15}$$

where $\bar{V}_2$ is the value of $\bar{V}$ attained at maximum altitude for $T/mg < ½$. From Eq. (14), it is seen that at this point $r_2/r_1 = 1/\bar{V}_2^2$. When $T/mg = ½$, the value of $\bar{V}_2$ given by Eq. (15) is 0, and the spiral path leading to maximum altitude at r_2 extends to infinity.

The angular rotation about the planet during the thrusting period is given in radians by

$$\theta_2 - \theta_1 = \pi/2 - \sin^{-1}\left\{\frac{(\bar{V}_2^2 + b)b/(1+b) + 2}{|\bar{V}_2^2 + b|\,[b^2/(1+b)^2 + 4/(1+b)]^{1/2}}\right\} \tag{16}$$

with $b = 2(T/mg - 1)$. [The derivation of Eq. (16) is presented in the appendix.] It is found that, for values of $T/mg < ½$, the angular rotation along the spiral path to maximum altitude is 180 deg. Likewise, with continued thrusting, the angular rotation back to the initial circular orbit along a symmetrical spiral path is also 180 deg.

Solution for Flight Time

The time of flight along a spiral trajectory from a circular orbit is found by combining Eqs. (6), (9), and (13) to obtain, with $\bar{V}_1 = 1$ and $\gamma_1 = 0$,

$$\sqrt{\mu/r_1^3}\, dt = \frac{2\bar{V}(\bar{V}^2 + b)\, d\bar{V}}{(1+b)^{3/2}[\bar{V}^2(\bar{V}^2 + b) - (1+b)]^{1/2}} \tag{17}$$

where, again, $b = 2(T/mg - 1)$. Then, letting $x = \bar{V}^2$ so that $dx = 2\bar{V}\,d\bar{V}$, Eq. (17) becomes

$$\sqrt{\mu/r_1^3}\,dt = \frac{(x+b)\,dx}{(1+b)^{3/2}[x(x+b)-(1+b)]^{1/2}} \tag{18}$$

which may be integrated according to standard formulas[6] to obtain

$$\sqrt{\mu/r_1^3}\,\Delta t = (1+b)^{-3/2}\Big\{[\bar{V}_2^2(\bar{V}_2^2+b)-(1+b)]^{1/2} + \frac{b}{2}\ln\Big[2[\bar{V}_2^2(\bar{V}_2^2+b)-(1+b)b]^{1/2}+2\bar{V}_2^2+b\Big] - \frac{b}{2}\ln(2+b)\Big\} \tag{19}$$

or, if $\bar{V}_2 = \sqrt{2}$,

$$\sqrt{\mu/r_1^3}\,\Delta t = (1+b)^{-3/2}\Big\{(3+b)^{1/2} + \frac{b}{2}\ln\Big[\frac{2(3+b)^{1/2}+4+b}{2+b}\Big]\Big\} \tag{20}$$

These solutions for flight time [Eqs. (19) and (20)] are valid for values of $T/mg > ½$. A different solution is required for values of $T/mg < ½$. It is found by putting Eq. (18) into the form

$$\sqrt{\mu/r_1^3}\,dt = \frac{(x+b)\,dx}{(1+b)[x^2(1+b)+xb(1+b)-(1+b)^2]^{1/2}} \tag{21}$$

Table 1 Flight conditions for nonescape from circular orbit using continuous vertical thrust and fixed thrust-to-weight ratio, $\gamma_2 = 0$

T/mg	b	$\bar{V}_2^2$	r_2/r_1	$\theta_2-\theta_1$ deg	$\sqrt{\mu/r_1^3}\Delta t$	m_1/m_1 for $I_{sp}=$ 250 s	500 s
0	−2.00	1.00	1.0000	180	3.14159	1.00000	1.00000
0.01	−1.98	0.98	1.0204		3.20587	0.89673	0.94696
0.05	−1.90	0.90	1.1111		3.49550	0.55198	0.74296
0.10	−1.80	0.80	1.2500		3.95146	0.26093	0.51081
0.20	−1.60	0.60	1.6667		5.40770	0.02529	0.15904
0.30	−1.40	0.40	2.5000	180	8.69276	0.00014	0.01188
0.40	−1.20	0.20	5.0000		21.07444	0.00000	0.00000
0.45	−1.10	0.10	10.0000		54.64024	0.00000	0.00000
0.49	−1.02	0.02	50.0000		566.4676	0.00000	0.00000
0.50	−1.00	0	—		—	0	0

$\mu = 1.4076\times10^{16}ft^3/s^2 = 0.3985\times10^6 km^3/s^2$ for Earth
$\mu = 468{,}000\times10^{16}ft^3/s^2 = 132{,}500\times10^6 km^3/s^2$ for sun

which also may be integrated according to standard formulas to obtain finally

$$\sqrt{\mu/r_1^3}\,\Delta t = \frac{b/(1+b)}{2[-(1+b)]^{1/2}}\Big[\pi/2 - \sin^{-1}\Big(\frac{2\bar{V}_2^2+b}{2+b}\Big)\Big] \tag{22}$$

Solution for Mass Loss

From the definition of specific impulse I_{sp} of the vehicle propulsion system (which is measured in seconds) is obtained

$$I_{sp} = \frac{\text{thrust in pounds}}{\text{pounds of propellant per second}} = T/\Big(-g\frac{dm}{dt}\Big) \tag{23}$$

so that

$$-I_{sp}\frac{dm}{m} = \Big(\frac{T}{mg}\Big)dt \tag{24}$$

which may be integrated, if T/mg is constant, to obtain

$$\ln(m_1/m_2) = (T/mg)\Delta t/I_{sp} \tag{25}$$

or

$$m_2/m_1 = \exp[-(T/mg)\Delta t/I_{sp}] \tag{26}$$

where the subscripts 1 and 2 denote conditions at the beginning and end, respectively, of the time interval Δt.

In the case of impulsive thrust ($\Delta t = 0$), Eq. (26) may be modified by assuming that the dimensionless velocity increment resulting from the vertical thrust impulse is described by Eq. (3) with γ and τ set equal to 0 so that

$$(T/mg)\Delta t = \sqrt{r/g)_1}\Delta\bar{V} = \sqrt{r_1^3/\mu}\Delta\bar{V} \tag{27}$$

where $\Delta\bar{V}$ is the magnitude of the vertical velocity vector that must be added (vectorially) to the initial velocity vector (whose magnitude $\bar{V}_1$ is 1) to obtain the resultant velocity vector (whose magnitude is $\bar{V}_2$). Thus, since $\bar{V}_2^2 = \bar{V}_1^2 + (\Delta\bar{V})^2$ for $\tau = 90$ deg, Eq. (26) becomes

$$m_2/m_1 = \exp[-\sqrt{r_1^3/\mu}(\bar{V}_2^2-1)^{1/2}/I_{sp}] \tag{28}$$

Numerical Results

Listings of numerical results obtained from solutions developed herein for continuous radial thrust applied to a vehicle in circular orbit are presented in Tables 1 and 2. The range of

Table 2 Flight conditions for escape from circular orbit using continuous vertical thrust and fixed thrust-to-weight ratio, $\bar{V}_2 = \sqrt{2}$

T/mg	b	γ_2, deg	r_2/r_1	$\theta_2-\theta_1$, deg	$\sqrt{\mu/r_1^3}\Delta t$	m_2/m_1 for $I_{sp}=$ 250 s	500 s
0.50	−1.0	90.000	——	180.000	——	0	0
0.55	−0.9	77.690	11.0000	130.760	21.68895	0.000000	0.000000
0.60	−0.8	73.221	6.0000	112.885	9.26310	0.000000	0.000079
0.70	−0.6	67.792	3.5000	91.169	4.30334	0.000036	0.005970
0.80	−0.4	64.341	2.6667	77.364	2.84522	0.000436	0.020869
0.90	−0.2	61.874	2.2500	67.498	2.14583	0.001407	0.037511
1.0	0	60.000	2.0000	60.000	1.73205	0.002770	0.052629
1.1	0.2	58.518	1.8333	54.071	1.45689	0.004302	0.065587
1.2	0.4	57.312	1.7143	49.249	1.25982	0.005857	0.076553
1.3	0.6	56.310	1.6250	45.240	1.11124	0.007360	0.085791
1.4	0.8	55.462	1.5556	41.850	0.99496	0.008774	0.093667
1.5	1.0	54.736	1.5000	38.942	0.90132	0.010085	0.100423
2.0	2.0	52.239	1.3333	28.955	0.61555	0.015211	0.123333
3.0	4.0	49.797	1.2000	19.188	0.37892	0.020963	0.144787
Impulsive thrust	0		1.0000	0	0	0.033373	0.182684

thrust-to-weight ratio covered in Table 1 is that for nonescape from orbit ($T/mg < ½$), whereas the range covered in Table 2 is a portion of that for escape ($T/mg > ½$). In both tables, values of vehicle mass ratio obtained using Eq. (26) are listed for two values of specific impulse, 250 s and 500 s. The lower value is typical of those for solid rocket propellants, whereas the higher value is about the maximum attainable using the best liquid rocket propellants (hydrogen and oxygen). For a vehicle in circular orbit around Earth at a radial distance of 4100 miles (6,598 km), the value of $\sqrt{r_1^3/\mu}$ is 850 s.

There is a direct application of the solution presented herein to the problem of interplanetary transfer from the orbit of Earth to that of Mars using solar radiation as a means of propulsion.[5] Since solar radiation pressure varies inversely as the square of radial distance from the sun, as does the acceleration of gravity due to the sun, the value of T/mg is constant when the area of the solar sail and the total mass are fixed. Thus, results of the kind presented in Table 1 apply directly. However, since the orbit of Mars is considerably more eccentric than the nearly-circular orbit of Earth, the value of r_2/r_1 can range from a minimum of about 1.38 to a maximum of about 1.66, depending on the particular epoch. In the average case, the value of r_2/r_1 is about 1.52, so that $\bar{V}_2^2 = 0.65789$, $T/mg = 0.17$, and $\sqrt{\mu/r_1^3}\Delta t = 4.8750$. For a vehicle in circular orbit around the sun at a radial distance equal to that of Earth (93,000,000 miles or about 150,000,000 km), the value of $\sqrt{r_1^3/\mu}$ is 5.0298×10^6 s or 58.215 days. The travel time, therefore, from the orbit of Earth to the midpoint (between aphelion and perihelion) on the orbit of Mars using solar radiation pressure for propulsion is found to be 58.215 d $\times$ 4.8750 = 283.8 days.

Conclusions

An analytic solution has been obtained for the effects of continuous radial (vertical) thrust on the orbital motion and mass loss of a vehicle initially in a circular orbit. It is found that, with continuous application of radial thrust equal in magnitude to a fixed fraction of the vehicle weight (as the product of vehicle mass and the ambient acceleration of gravity), the flight path can take one of two possible forms. If the value of thrust-to-weight ratio is greater than one-half, escape speed will eventually be reached along an unwinding spiral trajectory. If the value of this ratio is less than one-half, the vehicle will simply spiral out to a maximum altitude (apogee) and then return along a symmetrical trajectory to its initial position at the inception of thrusting. When the value of this ratio is one-half, the spiral path to maximum altitude extends to infinity. Formulas have been found for the orbital motion and time of flight along each trajectory and for mass loss due to expenditure of rocket propellant (based on the specific impulse of the propulsion system).

Appendix: Derivation of Angular Rotation

The flight-path angle γ is given by

$$\tan\gamma = \frac{dr}{r\,d\theta} \tag{A1}$$

with θ being the angle of rotation about the planet so that

$$d\theta = \frac{\cos\gamma}{\sin\gamma}\frac{dr}{r} \tag{A2}$$

and, consequently, using Eqs. (8) and (13) with $\bar{V}_1 = 1$ and $\gamma_1 = 0$,

$$d\theta = 2\bar{V}\left[\bar{V}^2(\bar{V}^2 + b)/(1+b) - 1\right]^{-1/2}\frac{d\bar{V}}{(\bar{V}^2 + b)} \tag{A3}$$

where $b = 2(T/mg - 1)$. Then, letting $x = \bar{V}^2$ so that $dx = 2\bar{V}\,d\bar{V}$, Eq. (A3) becomes

$$d\theta = \left[x(x+b)/(1+b) - 1\right]^{-1/2}\frac{dx}{x+b} \tag{A4}$$

which in turn, letting $y = x + b$ so that $dy = dx$, becomes

$$d\theta = \left[y(y-b)/(1+b) - 1\right]^{-1/2}\frac{dy}{y} = \left[\frac{y^2}{(1+b)} - \frac{by}{(1+b)} - 1\right]^{-1/2}\frac{dy}{y} \tag{A5}$$

This equation may be integrated according to a standard formula[6] to obtain

$$\theta = \sin^{-1}\left[\frac{-by/(1+b) - 2}{|y|\,[b^2/(1+b)^2 + 4/(1+b)]^{1/2}}\right] \tag{A6}$$

which, after applying the limits of integration, $y_1 = 1 + b$ and $y_2 = \bar{V}_2^2 + b$, yields finally

$$\theta_2 - \theta_1 \equiv \Delta\theta = \pi/2 - \sin^{-1}\left\{\frac{(\bar{V}_2^2 + b)b/(1+b) + 2}{|\bar{V}_2^2 + b|\,[b^2/(1+b)^2 + 4/(1+b)]^{1/2}}\right\} \tag{A7}$$

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Trajectory Design for Robotic Manipulators in Space Applications

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Introduction

THIS paper presents a trajectory design method for space robots that has been developed by the author. A subsequent paper that applies the technique to a practical robot has already been published.[1]

Robotic manipulators used in space applications are expected to operate under microgravity conditions.[2,3] Base reactions of a space manipulator are directly exerted on the supporting space structure, which would be typically a space vehicle (e.g., space shuttle) or a space station. It is desirable to

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